Annals of Fuzzy Mathematics and Informatics
Volume 1, No. 2, (April 2011), pp. 119- 131
ISSN 2093–9310
http://www.afmi.or.kr

On lacunary statistical convergence in intuitionistic fuzzy n-normed linear spaces

N. Thillaigovindan, S. Anita Shanthi, Young Bae Jun

Received 3 November 2010; Accepted 11 December 2010

Abstract. We introduce the concept of lacunary statistical convergence in intuitionistic fuzzy n-normed linear space. Some inclusion relations between the sets of statistically convergent and lacunary statistically convergent sequences are established in an intuitionistic fuzzy n-normed linear space. We also define lacunary statistical Cauchy sequence in an intuitionistic fuzzy n-normed linear space and prove that it is equivalent to lacunary statistically convergent sequence.

2010 AMS Classification: 40A05, 60B99

Keywords: Lacunary sequence, Lacunary statistical convergence, Lacunary statistical Cauchy sequence.

Corresponding Author: N. Thillaigovindan (thillai_n@sify.com)

1. Introduction

Motivated by the theory of n-normed linear space [12, 13, 16] and fuzzy normed linear space [1, 2, 4, 6, 7] the notions of fuzzy n-normed linear space [18] and intuitionistic fuzzy n-normed linear space [19] have been developed. The concept of statistical convergence for real number sequences was first introduced by Fast [5] and Schoneberg [22] independently. Later, it was further investigated from sequence point of view and linked with summability theory by Fridy [9], Salat [21] and many others. Fridy [10] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [3, 8, 11, 15, 20]. Karakus [14] et al. investigated the statistical convergence on intuitionistic fuzzy normed spaces and gave a characterization for statistically convergent sequences on these spaces. Mursaleen [17] et al. extended the results of Karakus from single sequence to double sequences on intuitionistic fuzzy normed spaces.

Our aim in this paper is to introduce the notions of lacunary statistical convergence and lacunary statistical Cauchy sequence in intuitionistic fuzzy n-normed linear spaces and establish some inclusion relations between statistical convergence and lacunary statistical convergence in intuitionistic fuzzy n-normed linear spaces.

2. Preliminaries

In this section we recall some useful definitions and results.

Definition 2.1 ([23]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$, for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.2 ([23]). A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-co-norm if $\diamond$satisfies the following conditions:

(i) $\diamond$ is commutative and associative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.3 ([19]). An intuitionistic fuzzy n-normed linear space or in short i-f-n-NLS is an object of the form

$$\mathbb{A} = \{(X, N(\mathbf{x}, t), M(\mathbf{x}, t)) \mid \mathbf{x} = (x_1, x_2, \cdots, x_n) \in X^n\}$$

where X is a linear space over a field $\mathbb{F}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$; N denotes the degree of membership and M denotes the degree of non-membership of $(x_1, x_2, \cdots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions:

(1) $N(\mathbf{x}, t) + M(\mathbf{x}, t) \leq 1$,
(2) $N(\mathbf{x}, t) > 0$,
(3) $N(x_1, x_2, \cdots, x_n, t) = 1$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
(4) $N(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$,
(5) $N(x_1, x_2, \cdots, cx_n, t) = N(x_1, x_2, \cdots, x_n, \frac{t}{|c|})$ if $c \neq 0$,$c \in \mathbb{F}$,
(6) $N(x_1, x_2, \cdots, x_n, s) * N(x_1, x_2, \cdots, x'_n, t) \leq N(x_1, x_2, \cdots, x_n + x'_n, s + t)$,
(7) $N(\mathbf{x}, \circ) : (0, \infty) \to [0,1]$ is continuous in t,
(8) $M(\mathbf{x}, t) > 0$,
(9) $M(x_1, x_2, \cdots, x_n, t) = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
(10) $M(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$,
(11) $M(x_1, x_2, \cdots, cx_n, t) = M\left(x_1, x_2, \cdots, x_n, \frac{t}{|c|}\right)$, if $c \neq 0$, $c \in \mathbb{F}$,
(12) $M(x_1, x_2, \cdots, x_n, s) \diamond M(x_1, x_2, \cdots, x'_n, t) \geq M(x_1, x_2, \cdots, x_n + x'_n, s + t)$,
(13) $M(\mathbf{x}, \circ) : (0, \infty) \to [0,1]$ is continuous in t.

Example 2.4 ([24]). Let $(X, ||\bullet, \cdots, \bullet||)$ be a n-normed linear space, where $X = \mathbb{R}$. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0,1]$,

$$N(x_1, x_2, \cdots, x_n, t) = e^{\frac{-\|x_1, x_2, \cdots, x_n\|}{t}}$$

and

$$M(x_1, x_2, \cdots, x_n, t) = 1 - e^{\frac{-\|x_1, x_2, \cdots, x_n\|}{t}}.$$

Then $\mathbb{A} = \{X, N(x_1, x_2, \cdots, x_n, t), M(x_1, x_2, \cdots, x_n, t) \mid (x_1, x_2, \cdots, x_n) \in X^n\}$ is an i-f-n-NLS.

Remark 2.5. For convenience we denote the intuitionistic fuzzy n-normed linear space by $\mathbb{A} = (X, N, M, *, \diamond)$.

Definition 2.6 ([2])**.** Let (U, N) be a fuzzy normed linear space. We define a set $B(x, \alpha, t)$ as $B(x, \alpha, t) = \{y : N(x - y, t) > 1 - \alpha\}$.

Definition 2.7 ([19])**.** A sequence $\{x_{n_k}\}$ in an i-f-n-NLS $\mathbb{A}$ is said to converge to $\xi \in X$ with respect to the intuitionistic fuzzy n-norm (N, M) if for each $\epsilon > 0$, $t > 0$ there exists a positive integer n_0 such that $N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) > 1 - \epsilon$ and $M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) < \epsilon$ for all $k \geq n_0$.

The element ξ is called the limit of the sequence $\{x_{n_k}\}$ with respect to the intuitionistic fuzzy n-norm (N, M) and is denoted as $(N, M) - \lim x_{n_k} = \xi$.

Definition 2.8 ([19])**.** A sequence $\{x_{n_k}\}$ in an i-f-n-NLS $\mathbb{A}$ is said to be Cauchy with respect to the intuitionistic fuzzy n-norm (N, M) if for each $\epsilon > 0$ and $t > 0$ there exists a positive integer m_0 such that $N(x_1, x_2, \cdots, x_{n-1}, x_{n_p} - x_{n_q}, t) > 1 - \epsilon$ and $M(x_1, x_2, \cdots, x_{n-1}, x_{n_p} - x_{n_q}, t) < \epsilon$ whenever $p, q \geq m_0$.

Definition 2.9 ([9])**.** A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

Definition 2.10 ([9])**.** For a lacunary sequence $\theta = \{k_r\}$, the number sequence $\{x_k\}$ is said to be lacunary statistically convergent to a number ξ provided that for each $\epsilon > 0$, $\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |x_k - \xi| \geq \epsilon\}| = 0$, where the vertical bar denotes the cardinality of the enclosed set. In this case we write $S_\theta - \lim_{k\to\infty} x_k = \xi$.

3. Lacunary statistical convergence in intuitionistic fuzzy n-normed linear space

Definition 3.1. Let $\mathbb{A}$ be an i-f-n-NLS. We define an open ball $B(x, r, t)$ with center x on the n^{th} coordinate of X^n and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in X : N(x_1, x_2, \cdots, x_{n-1}, x - y, t) > 1 - r \quad \text{and} \quad M(x_1, x_2, \cdots, x_{n-1}, x - y, t) < r\}$$

for $t > 0$.

Definition 3.2. Let $\mathbb{A}$ be an i-f-n-NLS. A sequence $\{x_{n_k}\}$ of elements in X is said to be statistically convergent to $\xi \in X$ with respect to the i-f-n-norm (N, M) if for each $\epsilon > 0$ and $t > 0$, there exists $p \in \mathbb{N}$ such that

$$\lim_{p\to\infty} \frac{1}{p}|\{k \leq p : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}| = 0.$$

The element ξ is called the statistical limit of the sequence $\{x_{n_k}\}$ with respect to the intuitionistic fuzzy n-norm (N, M) and is denoted as $S(N, M) - \lim x_{n_k} = \xi$ or $x_{n_k} \to \xi(S_{(N,M)})$.

Definition 3.3. Let $\mathbb{A}$ be an i-f-n-NLS and θ be a lacunary sequence. A sequence $\{x_{n_k}\}$ of elements in X is said to be lacunary statistically convergent to $\xi \in X$ with respect to the i-f-n-norm (N, M) if for each $\epsilon > 0$ and $t > 0$,

$$\lim_{r\to\infty} \tfrac{1}{h_r} |\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \le 1 - \epsilon \quad \text{and} \quad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \ge \epsilon\}| = 0.$$

The element ξ is called the lacunary statistical limit of the sequence $\{x_{n_k}\}$ with respect to the intuitionistic fuzzy n-norm (N, M) and is denoted as $S^\theta_{(N,M)} - \lim x_{n_k} = \xi$ or $x_{n_k} \to \xi(S^\theta_{(N,M)})$.

We denote by $S^\theta_{(N,M)}(X)$, the set of all lacunary statistically convergent sequences in an i-f-n-NLS $\mathbb{A}$.

Next we show that for any fixed $\theta, S^\theta_{(N,M)}-$limit is unique provided it exists.

Theorem 3.4. *Let $\mathbb{A}$ be an i-f-n-NLS and θ be a fixed lacunary sequence. If $\{x_{n_k}\}$ is a sequence in X such that $S^\theta_{(N,M)} - \lim x_{n_k} = \xi$ exists, then it is unique.*

Proof. Suppose that there exist elements ξ, η with $(\xi \ne \eta) \in X$ such that

$$S^\theta_{(N,M)} - \lim x_{n_k} = \xi \quad \text{and} \quad S^\theta_{(N,M)} - \lim x_{n_k} = \eta.$$

Let $\epsilon > 0$ be arbitrary. Choose $s > 0$ such that

$$(1 - s) * (1 - s) > 1 - \epsilon \text{ and } s \diamond s < \epsilon. \tag{3.1}$$

For $t > 0$, we take

$$A = \{k \in I_r \mid N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) > 1 - s, \quad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) < s\}$$

and

$$B = \{k \in I_r \mid N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \eta, t) > 1 - s, \quad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \eta, t) < s\}.$$

We shall first show that for $\xi \ne \eta$ and $t > 0$, $A \cap B = \phi$. For, if $m \in A \cap B$ then

$$\begin{aligned} &N(x_1, x_2, \cdots, x_{n-1}, \xi - \eta, t) \\ &\ge N(x_1, x_2, \cdots, x_{n-1}, x_m - \xi, \tfrac{t}{2}) * N(x_1, x_2, \cdots, x_{n-1}, x_m - \eta, \tfrac{t}{2}) \\ &> (1 - s) * (1 - s) > 1 - \epsilon. \end{aligned}$$

by (3.1). Since $\epsilon > 0$ is arbitrary we have $N(x_1, x_2, \cdots, x_{n-1}, \xi - \eta, t) = 1$ for every $t > 0$. Similarly $M(x_1, x_2, \cdots, x_{n-1}, \xi - \eta, t) = 0$ for every $t > 0$. This implies that $\xi - \eta = 0$, a contradiction to $\xi \ne \eta$. Thus $A \cap B = \phi$ and therefore $A \subset B^c$. Hence we have

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\{k\in I_r : N(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)>1-s \text{ and } M(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)<s\}\Big|$$
$$\leq \lim_{r\to\infty}\frac{1}{h_r}\Big|\{k\in I_r : N(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\eta,t)\leq 1-s \text{ and } M(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\eta,t)\geq s\}\Big|.$$

Since $S^{\theta}_{(N,M)}-\lim x_{n_k}=\eta$, it follows that

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\{k\in I_r : N(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)>1-s \text{ and } M(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)<s\}\Big|\leq 0.$$

Since this cannot be negative we have

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\{k\in I_r : N(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)>1-s \text{ and } M(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)<s\}\Big|= 0.$$

This contradicts the fact that $S^{\theta}_{(N,M)}-\lim x_{n_k}=\xi$. Hence $\xi=\eta$. □

Theorem 3.5. $S^{\theta}_{(N,M)}(X)$ *is a linear space.*

Proof. Let $\{x_{n_k}\}$ be a sequence in X.

(i) If $S^{\theta}_{(N,M)}-\lim x_{n_k}=\xi$ and $\alpha\neq 0\in\mathbb{R}$, then we need to prove that

$$S^{\theta}_{(N,M)}-\lim \alpha x_{n_k}=\alpha\,\xi.$$

Let $\xi>0$ and $t>0$. If we take

$$A=\{k\in I_r : N(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)>1-\epsilon \text{ and } M(x_1,x_2,\cdots,x_{n-1},x_{n_k}-\xi,t)<\epsilon\}$$

and

$$B=\{k\in I_r : N(x_1,x_2,\cdots,x_{n-1},\alpha x_{n_k}-\alpha\xi,t)>1-\epsilon \text{ and } M(x_1,x_2,\cdots,x_{n-1},\alpha x_{n_k}-\alpha\xi,t)<\epsilon\}.$$

Let $m\in A$. Then we have

$$\begin{aligned}N(x_1,x_2,\cdots,x_{n-1},\alpha x_m-\alpha\xi,t) &= N(x_1,x_2,\cdots,x_{n-1},x_m-\xi,\tfrac{t}{|\alpha|})\\ &\geq N(x_1,x_2,\cdots,x_{n-1},x_m-\xi,t)*N(0,\tfrac{t}{|\alpha|}-t)\\ &\geq N(x_1,x_2,\cdots,x_{n-1},x_m-\xi,t)*1\\ &\geq N(x_1,x_2,\cdots,x_{n-1},x_m-\xi,t)\\ &>1-\epsilon\end{aligned}$$

and

$$\begin{aligned}M(x_1,x_2,\cdots,x_{n-1},\alpha x_m-\alpha\xi,t) &= M(x_1,x_2,\cdots,x_{n-1},x_m-\xi,\tfrac{t}{|\alpha|})\\ &\leq M(x_1,x_2,\cdots,x_{n-1},x_m-\xi,t)\diamond M(0,\tfrac{t}{|\alpha|}-t)\\ &\leq M(x_1,x_2,\cdots,x_{n-1},x_m-\xi,t)\diamond 0\\ &\leq M(x_1,x_2,\cdots,x_{n-1},x_m-\xi,t)<\epsilon\end{aligned}$$

which implies that $m \in B$. Hence we have $A \subset B$ and therefore $B^c \subset A^c$. It follows that

$$\lim_{r\to\infty} \frac{1}{h_r} \big| \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, \alpha x_{n_k} - \alpha\xi, t) \le 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, \alpha x_{n_k} - \alpha\xi, t) \ge \epsilon\} \big| \le \lim_{r\to\infty} \frac{1}{h_r} \big| \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \le 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \ge \epsilon\} \big| .$$

Since $S^\theta_{(N,M)} - \lim x_{n_k} = \xi$, it follows that $S^\theta_{(N,M)} - \lim \alpha x_{n_k} = \alpha\xi$.

(ii) Let $\{x_{n_k}\}$ and $\{y_{n_k}\}$ be two sequences in X. If $S^\theta_{(N,M)} - \lim x_{n_k} = \xi$ and $S^\theta_{(N,M)} - \lim y_{n_k} = \eta$, then we have to prove that $S^\theta_{(N,M)} - \lim(x_{n_k} + y_{n_k}) = \xi + \eta$. Let $\epsilon > 0$ be given. Choose $s > 0$ as in (3.1). For $t > 0$, we define the following sets:

$$A = \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, (x_{n_k} + y_{n_k}) - (\xi + \eta), t) > 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, (x_{n_k} + y_{n_k}) - (\xi + \eta), t) < \epsilon\}$$

$$B = \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) > 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) < s\},$$

$$C = \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, y_{n_k} - \eta, t) > 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, y_{n_k} - \eta, t) < s\}.$$

Let $m \in B \cap C$. Then we have

$$\begin{aligned} &N(x_1, x_2, \cdots, x_{n-1}, (x_m + y_m) - (\xi + \eta), t) \\ &= N(x_1, x_2, \cdots, x_{n-1}, (x_m - \xi) + ((y_m - \eta), \tfrac{t}{2} + \tfrac{t}{2}) \\ &\ge N(x_1, x_2, \cdots, x_{n-1}, (x_m - \xi), \tfrac{t}{2}) * N(x_1, x_2, \cdots, x_{n-1}, ((y_m - \eta), \tfrac{t}{2}) \\ &> (1 - s) * (1 - s) > 1 - \epsilon \end{aligned}$$

by (3.1) and

$$\begin{aligned} &M(x_1, x_2, \cdots, x_{n-1}, (x_m + y_m) - (\xi + \eta), t) \\ &\le M(x_1, x_2, \cdots, x_{n-1}, x_m - \xi, \tfrac{t}{2}) \diamond M(x_1, x_2, \cdots, x_{n-1}, y_m - \eta, \tfrac{t}{2}) \\ &< s \diamond s < \epsilon. \end{aligned}$$

This shows that $m \in A$ and hence $(B \cap C) \subset A$. Therefore we have $A^c \subset (B^c \cup C^c)$. It follows that

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{h_r} \big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, (x_{n_k} + y_{n_k}) - (\xi + \eta), t) \le 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, (x_{n_k} + y_{n_k}) - (\xi + \eta), t) \ge \epsilon\}\big| \\ &\le \lim_{r\to\infty} \frac{1}{h_r} \big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \le 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \ge s\}\big| \\ &\quad + \lim_{r\to\infty} \frac{1}{h_r} \big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, y_{n_k} - \eta, t) \le 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, y_{n_k} - \eta, t) \ge s\}\big|. \end{aligned}$$

Since $S^{\theta}_{(N,M)} - \lim x_{n_k} = \xi$ and $S^{\theta}_{(N,M)} - \lim y_{n_k} = \eta$ we have

$$S^{\theta}_{(N,M)} - \lim(x_{n_k} + y_{n_k}) = \xi + \eta.$$

This completes the proof. □

Theorem 3.6. *Let $\mathbb{A}$ be an i-f-n-NLS. For any lacunary sequence θ, $S_{(N,M)}(X) \subset S^{\theta}_{(N,M)}(X)$ if and only if $\liminf_r q_r > 1$.*

Proof. Sufficient part: Suppose that $\liminf_r q_r > 1$. Then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r which implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$. If $\{x_{n_k}\}$ is statistically convergent to ξ with respect to i-f-n-norm (N, M), then for each $\epsilon > 0, t > 0$ and sufficiently large r, we have

$$\begin{aligned} &\tfrac{\delta}{1+\delta}\tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and} \\ &\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big| \\ &\leq \tfrac{1}{k_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and} \\ &\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big| \\ &\leq \tfrac{1}{k_r}\big|\{k \leq k_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and} \\ &\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big|. \end{aligned}$$

It follows that $x_{n_k} \to \xi(s^{\theta}_{(N,M)})$. Hence $S_{(N,M)}(X) \subset S^{\theta}_{(N,M)}(X)$.

Necessary part: Suppose that $\liminf_r q_r = 1$. Then we can select a subsequence $\{k_{r(j)}\}$ of the lacunary sequence θ such that $\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j}$ and $\frac{k_{r(j)-1}}{k_{r(j-1)}} > j$, where $r(j) \geq r(j-1) + 2$. Let $\xi \neq 0 \in X$. We define a sequence $\{x_{n_k}\}$ as follows:

$$x_{n_k} = \begin{cases} \xi, & \text{if} \quad k \in I_{r(j)} \quad \text{for some} \quad j = 1, 2, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

We shall show that $\{x_{n_k}\}$ is statistically convergent to ξ with respect to the i-f-n-norm (N,M). Let $\epsilon > 0$ and $t > 0$. Choose $\epsilon_1 \in (0, 1)$ such that $B(0, \epsilon_1, t) \subset B(0.\epsilon, t)$ and $\xi \notin B(0, \epsilon, t)$. Also for each p we can find a positive number j_p such that

$k_{r(j_p)} < p \leq k_{r(j_p)+1}$. Then we have

$$\begin{aligned}
&\tfrac{1}{p}\big|\{k \leq p : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon, \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big| \\
&\leq \tfrac{1}{k_{r(j_p)}}\big|\{k \leq p : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon_1, \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon_1\}\big| \\
&\leq \tfrac{1}{k_{r(j_p)}}\big[\big|\{k \leq k_{r(j_p)} : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon_1, \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon_1\}\big| \\
&\quad + \big|\{k_{r_{(j_p)}} < k \leq p : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon_1, \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon_1\}\big|\big] \\
&\leq \tfrac{1}{k_{r(j_p)}}\big|\{k \leq k_{r(j_p)} : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon_1, \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon_1\}\big| \\
&\quad + \tfrac{1}{k_{r(j_p)}}(k_{r(j_p)+1} - k_{r(j_p)}) \\
&< \tfrac{1}{j_p} + \tfrac{1}{j_p+1} + 1 - 1 = \tfrac{1}{j_p} + \tfrac{1}{j_p+1}
\end{aligned}$$

for each p. It follows that $S(N, M) - \lim x_{n_k} = \xi$. Next we shall show that $\{x_{n_k}\}$ is not lacunary statistically convergent with respect to the i-f-n-norm(N, M). Since $\xi \neq 0$ we choose $\epsilon > 0$ such that $\xi \notin B(0, \epsilon, t)$ for $t > 0$. Thus

$$\begin{aligned}
&\lim_{j\to\infty} \tfrac{1}{h_{r(j)}}\big|\{k_{r(j)-1} < k \leq k_{r(j)} : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - 0, t) \leq 1 - \epsilon, \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - 0, t) \geq \epsilon\}\big| \\
&= \lim_{j\to\infty} \tfrac{1}{h_{r(j)}}(k_{r(j)} - k_{r(j-1)}) = \lim_{j\to\infty} \tfrac{1}{h_{r(j)}}(h_{r(j)}) = 1
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\substack{r\to\infty \\ r\neq r(j), j=1,2,\cdots}} \tfrac{1}{h_r}\big|\{k_{r-1} < k \leq k_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon, \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big| = 1 \neq 0.
\end{aligned}$$

Hence neither ξ nor 0 can be lacunary statistical limit of the sequence $\{x_{n_k}\}$ with respect to the intuitionistic fuzzy n-norm (N, M). No other point of X can be lacunary statistical limit of the sequence as well. Hence $\{x_{n_k}\} \notin S^{\theta}_{(N,M)}(X)$ completing the proof. □

The following example establishes that lacunary statistical convergence need not imply statistical convergence.

Example 3.7. Let $(X, ||\bullet, \cdots, \bullet||)$ be a n-normed linear space, where $X = \mathbb{R}$. Define $a * b = ab$ and $a \diamond b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$,

$$N(x_1, x_2, \cdots, x_{n_k}, t) = \tfrac{t}{t+\|x_1, x_2, \cdots, x_{n_k}\|}$$

and

$$M(x_1, x_2, \cdots, x_{n_k}, t) = \tfrac{\|x_1, x_2, \cdots, x_{n_k}\|}{t+\|x_1, x_2, \cdots, x_{n_k}\|}.$$

Then $\mathbb{A} = (X, N, M, *, \diamond)$ is an i-f-n-NLS. We define a sequence $\{x_{n_k}\}$ by

$$x_{n_k} = \begin{cases} nk & \text{for} \quad k_r - (\sqrt{h_r}) + 1 \leq k \leq k_r, r \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

For $\epsilon > 0$ and $t > 0$,

$$K_r(\epsilon, t) = \{k \in \mathbb{N} : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k}, t) \leq 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k}, t) \geq \epsilon\}.$$

Then

$$\begin{aligned} K_r(\epsilon, t) &= \{k \in \mathbb{N} : \tfrac{t}{t + \|x_1, x_2, \cdots, x_{n-1}, x_{n_k}\|} \leq 1 - \epsilon \text{ and } \tfrac{\|x_1, x_2, \cdots, x_{n_k}\|}{t + \|x_1, x_2, \cdots, x_{n-1}, x_{n_k}\|} \geq \epsilon\} \\ &= \{k \in \mathbb{N} : \|x_1, x_2, \cdots, x_{n-1}, x_{n_k}\| \geq \tfrac{\epsilon t}{1-\epsilon} > 0\} \\ &= \{k \in \mathbb{N} : x_{n_k} = nk\} \\ &= \{k \in \mathbb{N} : k_r - (\sqrt{h_r}) + 1 \leq k \leq k_r, r \in \mathbb{N}\} \end{aligned}$$

and so, we get

$$\tfrac{1}{h_r}|K_r(\epsilon, t)| = \tfrac{1}{h_r}|\{k \in \mathbb{N} : k_r - (\sqrt{h_r}) + 1 \leq k \leq k_r, r \in \mathbb{N}\}| \leq \tfrac{\sqrt{h_r}}{h_r}$$
$$\Rightarrow \lim_{r \to \infty} \tfrac{1}{h_r}|K_r(\epsilon, t)| = 0$$
$$\Rightarrow x_{n_k} \to 0 \ (S^{\theta}_{(N,M)}).$$

On the other hand $x_{n_k} \nrightarrow 0 \ (S_{(N,M)})$, since

$$\begin{aligned} N(x_1, x_2, \cdots, x_{n-1}, x_{n_k}, t) &= \tfrac{t}{t + \|x_1, x_2, \cdots, x_{n-1}, x_{n_k}\|} \\ &= \begin{cases} \frac{t}{t+nk}, & \text{for} \quad k_r - (\sqrt{h_r}) + 1 \leq k \leq k_r, r \in \mathbb{N} \\ 1, & \text{otherwise} \end{cases} \\ &\leq 1 \end{aligned}$$

and

$$\begin{aligned} M(x_1, x_2, \cdots, x_{n-1}, x_{n_k}, t) &= \tfrac{\|x_1, x_2, \cdots, x_{n_k}\|}{t + \|x_1, x_2, \cdots, x_{n_k}\|} \\ &= \begin{cases} \frac{nk}{t+nk}, & \text{for} \quad k_r - (\sqrt{h_r}) + 1 \leq k \leq k_r, r \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \\ &\geq 0. \end{aligned}$$

Hence $x_{n_k} \nrightarrow 0 \ (S_{(N,M)})$.

Theorem 3.8. *Let $\mathbb{A}$ be an i-f-n-NLS. For any lacunary sequence θ, $S^{\theta}_{(N,M)}(X) \subset S_{(N,M)}(X)$ if and only if $\limsup_r q_r < \infty$.*

Proof. Sufficient part: If $\limsup_r q_r < \infty$, then there is a $H > 0$ such that $q_r < H$ for all r. Suppose that $x_{n_k} \to \xi \ (S^{\theta}_{(N,M)})$, and let

$$P_r = \big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big|.$$

By definition of a lacunary statistical convergent sequence, there is a positive number r_0 such that

$$\frac{P_r}{h_r} < \epsilon \text{ forall } r > r_0. \tag{3.2}$$

Now let $K = \max\{P_r : 1 \leq r \leq r_0\}$ and p be any integer satisfying $k_{r-1} < p \leq k_r$. Then we have

$$\begin{aligned}
&\tfrac{1}{p}\big|\{k \leq p : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and} \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big| \\
&\leq \tfrac{1}{k_{r-1}}\big|\{k \leq k_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and} \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big| \\
&\leq \tfrac{1}{k_{r-1}}\{P_1 + P_2 + \cdots + P_{r_0} + P_{r_0+1} + \cdots + P_r\} \\
&\leq \tfrac{K}{k_{r-1}}\, r_0 + \tfrac{1}{k_{r-1}}\{h_{r_0+1}\, \tfrac{P_{r_0+1}}{h_{r_0+1}} + \cdots + h_r \tfrac{P_r}{h_r}\} \\
&\leq \tfrac{r_0\, K}{k_{r-1}} + \tfrac{1}{k_{r-1}}\left(\sup_{r > r_0} \tfrac{P_r}{h_r}\right)\{h_{r_0+1} + \cdots + h_r\} \\
&\leq \tfrac{r_0\, K}{k_{r-1}} + \epsilon \tfrac{k_r - k_{r_0}}{k_{r-1}} \qquad\qquad \text{(by (3.2))} \\
&\leq \tfrac{r_0\, K}{k_{r-1}} + \epsilon\, q_r \leq \tfrac{r_0\, K}{k_{r-1}} + \epsilon\, H,
\end{aligned}$$

and so $\{x_{n_k}\}$ is statistically convergent. Hence $S^{\theta}_{(N,M)}(X) \subset S_{(N,M)}(X)$.

Necessary part: Suppose that $\limsup_r q_r = \infty$. Let $\xi \neq 0 \in X$. Select a subsequence $\{k_{r(j)}\}$ of the lacunary sequence $\theta = \{k_r\}$ such that $q_{r(j)} > j,\ k_{r(j)} > j + 3$. Define a sequence $\{x_{n_k}\}$ as follows:

$$x_{n_k} = \begin{cases} \xi, & \text{if } \quad k_{r(j)-1} < k \leq 2k_{r(j)-1} \text{ for some } j = 1, 2, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

Since $\xi \neq 0$ we can choose $\epsilon > 0$ such that $\xi \notin B(0, \epsilon, t)$ for $t > 0$. Now for $j > 1$,

$$\begin{aligned}
&\tfrac{1}{h_{r(j)}}\big|\{k \leq k_{r(j)} : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - 0, t) \leq 1 - \epsilon \text{ and} \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - 0, t) \geq \epsilon\}\big| \\
&< \tfrac{1}{h_{r(j)}}\,(k_{r(j)-1}) < \tfrac{1}{(k_{r(j)} - k_{r(j)-1})}\,(k_{r(j)-1}) < \tfrac{1}{j-1}.
\end{aligned}$$

Therefore we have $\{x_{n_k}\} \in S^{\theta}_{(N,M)}(X)$. But $\{x_{n_k}\} \notin S_{(N,M)}(X)$. For

$$\begin{aligned}
&\tfrac{1}{2k_{r(j)-1}}\big|\{k \leq 2k_{r(j)-1} : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - 0, t) \leq 1 - \epsilon \text{ and} \\
&\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - 0, t) \geq \epsilon\}\big| \\
&= \tfrac{1}{2k_{r(j)}-1}\{k_{r(1)-1} + k_{r(2)-1} + \cdots + k_{r(j)-1}\} > \tfrac{1}{2}.
\end{aligned}$$

This shows that $\{x_{n_k}\}$ cannot be statistically convergent with respect to the intuitionistic fuzzy n-norm (N, M). □

Theorems 3.6 and 3.8 immediately give the following corollary.

Corollary 3.9. *Let* $\mathbb{A}$ *be an i-f-n-NLS. For any lacunary sequence* θ, $S^{\theta}_{(N,M)}(X) = S_{(N,M)}(X)$ *if and only if* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$.

Definition 3.10. Let $\mathbb{A}$ be an i-f-n-NLS and θ be a lacunary sequence. A sequence $\{x_{n_k}\}$ in X is said to be lacunary-θ-statistically Cauchy provided there is a subsequence $\{x_{n_{k'(r)}}\}$ of the sequence $\{x_{n_k}\}$ such that $k'(r) \in I_r$ for each $r, (N,M) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$ and for each $\epsilon > 0$ and $t > 0$,

$$\lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) \leq 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) \geq \epsilon\}\big| = 0.$$

Theorem 3.11. *Let* $\mathbb{A}$ *be an i-f-n-NLS and* θ *be lacunary sequence. A sequence* $\{x_{n_k}\}$ *in* X *is lacunary-*θ*-statistically convergent if and only if it is lacunary-*θ*-statistically Cauchy.*

Proof. We first assume that $S^{\theta}_{(N,M)} - \lim x_{n_k} = \xi$. For $t > 0$ and $j \in \mathbb{N}$, let

$$K(j,t) = \{k \in \mathbb{N} : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) > 1 - \tfrac{1}{j} \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) < \tfrac{1}{j}\}.$$

Then we have the following:

(i) $K(j+1,t) \subset K(j,t)$ and
(ii) $\frac{|K(j,\,t)\,\cap\, I_r|}{h_r} \to 1$ as $r \to \infty$.

This implies that we can choose a positive integer $m(1)$ such that for $r \geq m(1)$, we have $\frac{|K(1,\,t)\,\cap\, I_r|}{h_r} > 0$. i.e., $K(1,t) \cap I_r \neq \varnothing$. Next we can choose $m(2) > m(1)$ so that $r \geq m(2)$ implies $K(2,t) \cap I_r \neq \varnothing$. Thus for each r satisfying $m(1) \leq r \leq m(2)$, choose $k'(r) \in I_r$ such that $k'(r) \in I_r \cap K(1,t)$, i.e.,

$$N(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) > 0$$

and

$$M(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) < 1.$$

In general, we can choose $m(p+1) > m(p)$ such that

$$r > m(p+1) \quad \text{implies} \quad I_r \cap K(p+1,t) \neq \varnothing.$$

Thus for all r satisfying $m(p) \leq r \leq m(p+1)$, choose $k'(r) \in I_r \cap K(p,t)$, i.e.,

$$N(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) > 1 - \tfrac{1}{p} \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) < \tfrac{1}{p}. \tag{3.3}$$

Thus $k'(r) \in I_r$ for each r together with (3.3) implies that $(N,M) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$. For $\epsilon > 0$, choose $s > 0$ such that $(1-s) * (1-s) > 1 - \epsilon$ and $s \diamond s < \epsilon$. For $t > 0$, if we take

$$A = \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) > 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) < \epsilon\}$$

$$B = \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) > 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) < s\}$$

$$C = \{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) > 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) < s\},$$

then we find that $(B \cap C) \subset A$ and therefore $A^c \subset (B^c \cup C^c)$.
Thus we have

$$\begin{aligned} &\lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) \leq 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) \geq \epsilon\}\big| \\ &\leq \lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq s\}\big| \\ &\quad + \lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) \leq 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) \geq s\}\big|. \end{aligned}$$

Since $S^{\theta}_{(N,M)} - \lim x_{n_k} = \xi$ and $(N, M) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$ it follows that $\{x_{n_k}\}$ is $S^{\theta}_{(N,M)}$- Cauchy.

Conversely, suppose that $\{x_{n_k}\}$ is a lacunary-θ-statistically Cauchy sequence with respect to the i-f-n-norm (N, M). By definition there is a subsequence $\{x_{n_{k'(r)}}\}$ of the sequence $\{x_{n_k}\}$, such that $k'(r) \in I_r$ and for each $r, (N, M) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$ and for each $\epsilon > 0$ and $t > 0$,

$$\lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) \leq 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) \geq \epsilon\}\big| = 0. \tag{3.4}$$

As before we have the following inequality

$$\begin{aligned} &\lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \leq 1 - \epsilon \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - \xi, t) \geq \epsilon\}\big| \\ &\leq \lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}}, t) \leq 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_k} - x_{n_{k'(r)}} - \xi, t) \geq s\}\big| \\ &\quad + \lim_{r\to\infty} \tfrac{1}{h_r}\big|\{k \in I_r : N(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) \leq 1 - s \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_{n_{k'(r)}} - \xi, t) \geq s\}\big|. \end{aligned}$$

Since $(N, M) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$ it follows from (3.4) that $x_{n_k} \to \xi(S^{\theta}_{(N,M)})$. □

Acknowledgements. The authors are thankful to the reviewers for their valuable comments and suggestions towards the quality improvement of the paper.

References

[1] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (2003) 687–705.

[2] T. Bag and S. K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems 151 (2005) 513–547.

[3] H. Cakalli, Lacunary statistical convergence in topological groups, Indian J. Pure. Appl. Math. 26 (1995) 113–119.

[4] S. C. Chang and J. N. Mordesen, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Cal. Math. Soc. 86 (1994) 429–436.

[5] H. Fast, Surla convergence statistique, Colloq. Math. 2 (1951) 241–244.

[6] C. Felbin, The completion of fuzzy normed linear space, J. Math. Anal. Appl. 174 (1993) 428–440.

[7] C. K. Felbin, Finite dimensional fuzzy normed linear spaces II. p-adic analysis, summability theory, fuzzy analysis and applications (INCOPASFA) (Chennai, 1998), J. Anal. 7 (1999) 117–131.

[8] A. R. Freedman, J. J. Sember and M. Raphael, Same Cesaro type summability spaces, Proc. London. Math. Soc. 37 (1978) 508–520.

[9] J. A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.

[10] J. A. Fridy and C. Orhan, Lacunary statistical convergence, Pacific.J. Math. 160 (1993) 43–51.

[11] J. A. Fridy and C. Orhan, Lacunary statistical summability, J. Math. Anal. Appl. 173 (1993) 497–504.

[12] S. Gahler, Linear-2-normierte Raume, Math. Nachr. 28 (1965) 1–43.

[13] Hendra Gunawan and M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. 27 (2001) 631–639.

[14] S. Karakus, K. Demirci and O. Duman, Statisitical convegence on intuitionistic fuzzy normed spaces, Chaos Solitons and Fractals 35 (2008) 763–769.

[15] J. S. Kwon and H. T. Shim, Remark on lacunary statistical convergence of fuzzy numbers, Fuzzy sets and systems 123 (2001) 85–88.

[16] R. Malceski, Strong n-convex n-normed spaces, Mat. Bilten 21 (1997) 81–102.

[17] M. Mursaleen and S. A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Choas Solitons and Fractals 41 (2009) 2414–2421.

[18] Al. Narayanan and S. Vijayabalaji, Fuzzy n-normed linear space, Int. J. Math. Math. Sci. 24 (2005) 3963–3977.

[19] Al. Narayanan, S. Vijayabalaji and N. Thillaigovindan, Intuitionisitic fuzzy bounded linear operators, Iran. J. Fuzzy Systems 4 (2007) 89–101.

[20] F. Nuray, Lacunary statistical convergence of sequences of fuzzy numbers, Fuzzy sets and systems 99 (1998) 353–355.

[21] T. Salat, On statistical convergence of real numbers, Math. Slovaca. 30 (1980) 139–150.

[22] I. J. Schoneberg, The integrability of certain function and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.

[23] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 314–334.

[24] N. Thillaigovindan and S. Anita Shanthi, On intuitionistic fuzzy compact linear operators, Int. Review Fuzzy Math. (to appear).

N. Thillaigovindan (thillai_n@sify.com) – Department of Mathematics, Annamalai University, Annamalainagar 608002, Tamilnadu, India.

S. Anita Shanthi (shanthi.anita@yahoo.com) – Department of Mathematics, Annamalai University, Annamalainagar 608002, Tamilnadu, India.

Young Bae Jun (skywine@gmail.com) – Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea.